Supporting Secondary Teachers' Conceptions of Mathematical Modeling

Michelle Cirillo
John A. Pelesko
Department of Mathematical Sciences
University of Delaware

NCSM Conference
April 13, 2016

University of Delaware | College of Arts & Sciences

COMMON CORE STATE STANDARDS FOR

Mathematics

COMMON CORE
STATE STANDARDS INITIATIVE
PREPARING AMERICA'S STUDENTS FOR COLLEGE & CAREER

High School — Number and Quantity

High School — Algebra

High School — Functions

High School — Modeling

Mathematics | High School—Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

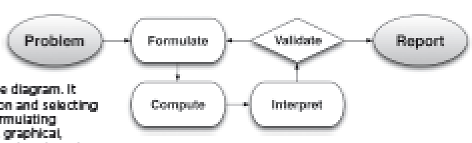
A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.



The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it

is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

Modeling Standards *Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (*).*

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Middle School Teacher Preparation

Additional Math Courses (9 credits)

Table 1. Mathematics Courses Recommended in MET II for Middle School Teachers that Are Required in Programs Preparing Middle Grades Teachers

Course	Number of Programs	Percentage of Programs (n=64)
Calculus	63	98%
Probability and Statistics	58	91%
Discrete Mathematics	45	70%
Number Theory	22	34%
History of Mathematics	12	19%
Mathematical Modeling	9	14%

Newton, Maeda, Alexander, Senk (2014) *Notices of the AMS*

High School Teacher Preparation

Additional Math Courses (18 credits)

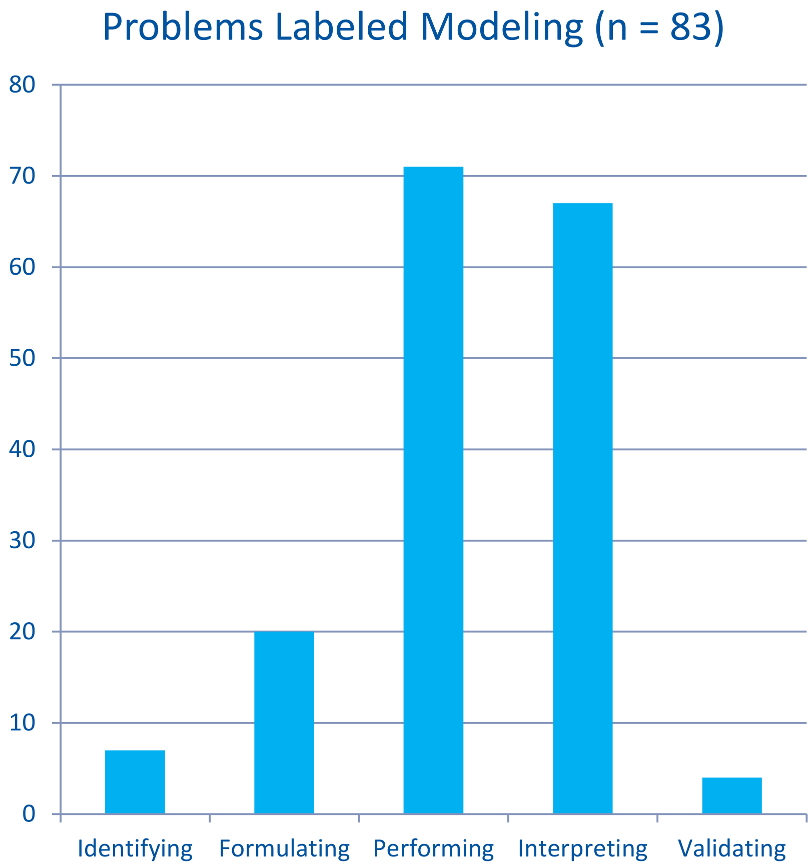
Table 2. Advanced Mathematics Courses Required in Programs Preparing High School Teachers

Course	Number of Programs	Percentage of Programs (n=78)
Geometry	70	90%
Abstract Algebra	61	78%
Discrete Mathematics	52	67%
Reasoning and Proof	47	60%
Mathematics Capstone Course	36	46%
Differential Equations	27	35%
Number Theory	24	31%
Real Analysis	23	29%
History of Mathematics	14	18%
Mathematical Modeling	12	15%

Newton, Maeda, Alexander, Senk (2014) *Notices of the AMS*

University of Delaware | College of Arts & Sciences

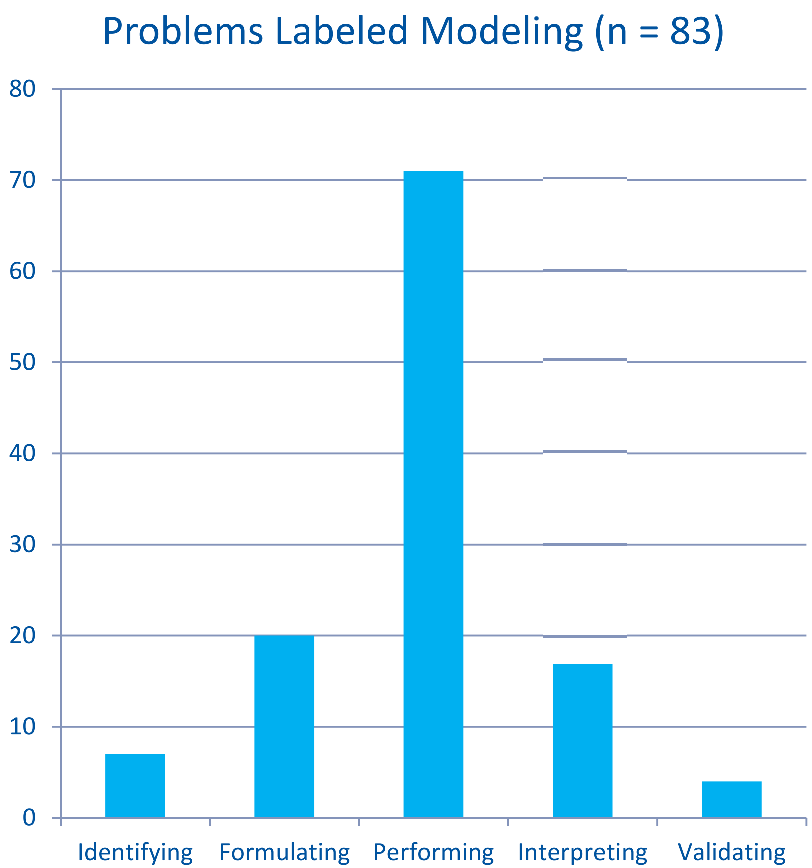
Dan Meyer's Textbook Analysis (NCSM, 2014)



Algebra I & Geometry

- *Identifying* variables in the situation and selecting those that represent essential features
- *Formulating* a model by creating and selecting [appropriate] representations
- *Analyzing* and performing operations on these relationships to draw conclusions
- *Interpreting* results of the mathematics in terms of the original situation
- *Validating* the conclusions by comparing them with the situation and then either improving the model, or if it is acceptable,
- *Reporting* on the conclusions and the reasoning behind them.

Dan Meyer's Textbook Analysis
(NCSM, 2014)



Algebra I & Geometry

- *Identifying* variables in the situation and selecting those that represent essential features
- *Formulating* a model by creating and selecting [appropriate] representations
- *Analyzing* and performing operations on these relationships to draw conclusions
- *Interpreting* results of the mathematics in terms of the original situation
- *Validating* the conclusions by comparing them with the situation and then either improving the model, or if it is acceptable,
- *Reporting* on the conclusions and the reasoning behind them.

What is new and different about mathematical modeling tasks?

- Not well-formulated

"Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process."

- More open-ended (involving creativity & assumptions)
- Require decisions to be made about what to keep and ignore

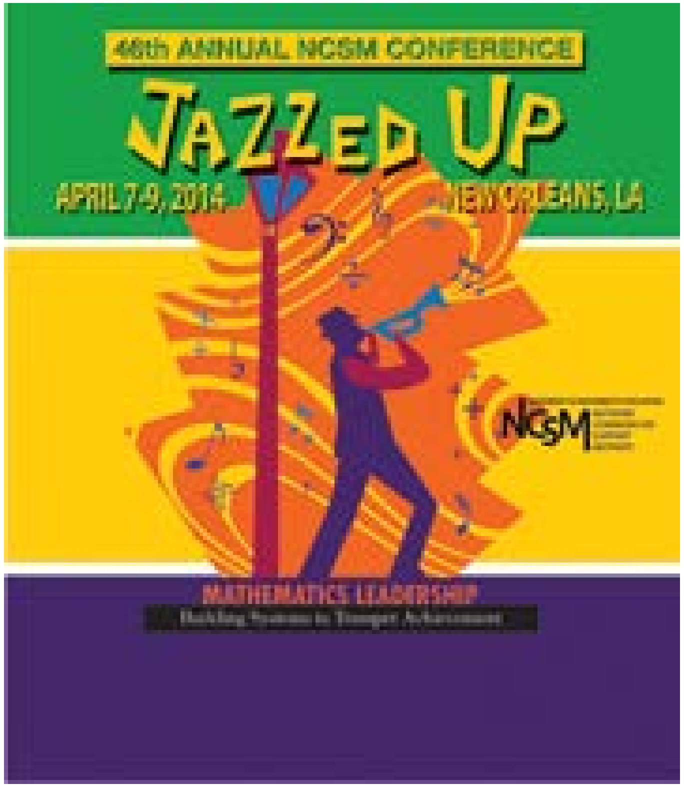
"Choices, assumptions, and approximations are present throughout this cycle."

- No one "correct" answer
- Requires time

"substantial investment in time" (NCTM, 2000)

What is new and different about mathematical modeling tasks?

"These Standards are not intended to be new names for old ways of doing business. They are a call to take the next step." (CCSSM, p. 5)



How many of these would fit in the Super Dome?

Volume of Super Dome is about 130,000,000 ft^3

Volume of a ping-pong ball is about $\frac{4}{1750}$ ft^3

One way to reason

$$Number\ of\ ping\ pong\ balls \approx \frac{Volume\ of\ Superdome}{Volume\ of\ single\ ball}$$

$$Number\ of\ ping\ pong\ balls \approx \frac{130000000}{4/1750} \approx 56{,}875{,}000{,}000$$

Reaction – Is this mathematical modeling? Or perhaps, Is this a modeling problem?

Q: Is our ping-pong ball problem mathematical modeling?

A: No, but it could be.

Q: How? How do we know?

6. A book shelf is 3 1/2 feet long. Each book on the shelf is 5/8 inches wide. How many books will fit on the shelf?
   Answer:
   Step 1: Convert the length of the bookshelf into inches:
   3 1/2 feet * 12 inches/foot = 42 inches
   Step 2: Divide
   42 ÷ 5/8 = 42 x 8/5 = 336/5 = 67 1/5,
   67 books will fit on the shelf.

Q: Is our ping-pong ball problem mathematical modeling?

A: No, but it could be.

Q: How? How do we know?

3. The serving size for the granola that Ted likes to eat for breakfast is 3/4 cup. How many servings are there in a box that holds 13 cups?
   Answer:
   13/1 ÷ 3/4 = 13/1 x 4/3 = 52/3, which simplifies to 17 1/3.
   There are 17 1/3 servings.

Q: Is our ping-pong ball problem mathematical modeling?

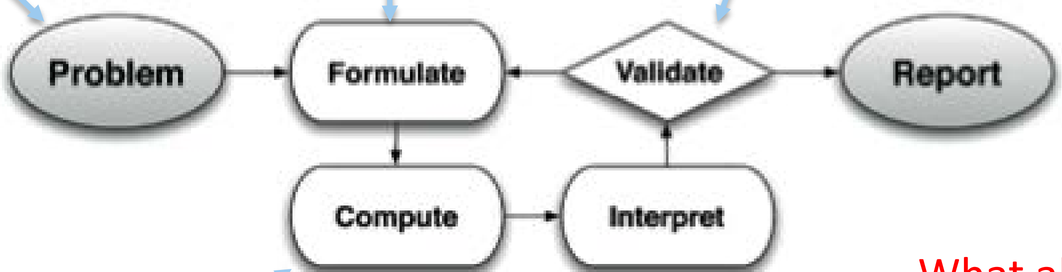
The volume of the Superdome is 130,000,000 ft^3. How many ping-pong balls could fit in the Superdome if the volume of each ping-pong ball is 4/1750 ft^3?

Mathematical modeling is a ***process***, a way of approaching problems, a tool for understanding phenomena in the real-world, a way for uncovering relationships.

How many ping-pong balls would fit in the Superdome?

Divide the two volumes

?



Divide?

Look at my answer?

What about iterating? Why would I ever do that for a problem like this?

Q: Is our ping-pong ball problem mathematical modeling?

Strategy #2 – When faced with a question like this, have your teachers and students examine their own thinking against the modeling process. If it feels forced, or if there is no clear reason for ever iterating, no clear decision-making, or assumptions, it's likely that you haven't engaged your students in mathematical modeling.

Cirillo & Pelesko (2016) NCSM Modeling Session 3214 www.modelwithmathematics.com

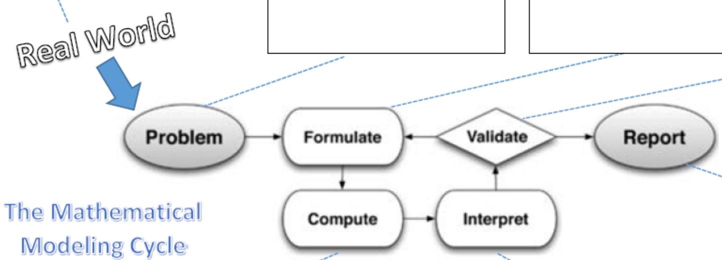
What **assumptions** are you making?

Real World - What aspect of the real world are you trying to model? (What is the "why" or "how" question you are trying to answer?)

Problem - Identifying variables in the situation and selecting those that represent essential features

Formulate – formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe the relationships between the variables

Validate – validating the conclusions by comparing them with the situation



What **decisions** did you make?

Report – reporting on the conclusions and the reasoning behind them.

~~**Compute**~~ **Analyze** – analyzing and performing operations on these relationships to draw conclusions

Interpret – interpreting the results of the mathematics in terms of the original situation

Choices, assumptions, and approximations are present throughout this cycle!

Q: Is our ping-pong ball problem mathematical modeling?

A: No, *but it could be.*

$$N = \#\ of\ balls\ that\ can\ fit\ in\ a\ given\ volume$$

$$N = N(Vs, Vb)$$

$$N = \frac{V_s}{V_b}$$

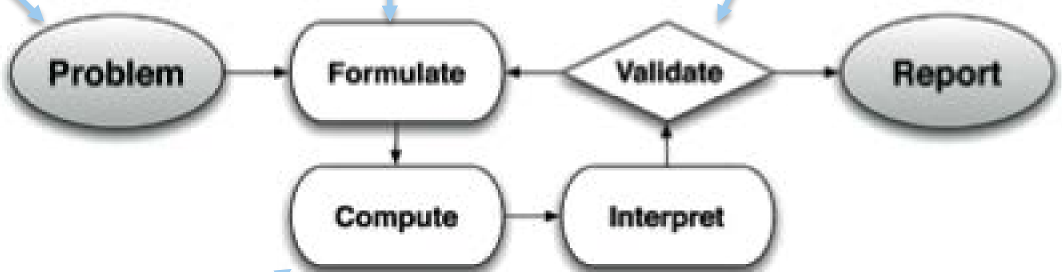
$$N = c\frac{V_s}{V_b}$$

Mathematical modeling is both about the choice of problem ***AND the approach you take to solving that problem.***

How many ping-pong balls would fit in the Superdome?

Number is a function of two volumes

Not the Superdome, but a shoe box.

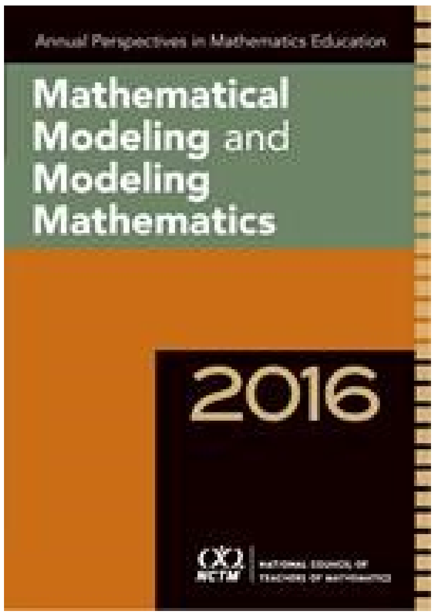


You start to recognize your assumptions, this lets you think them through carefully!

Really about *analyzing*!

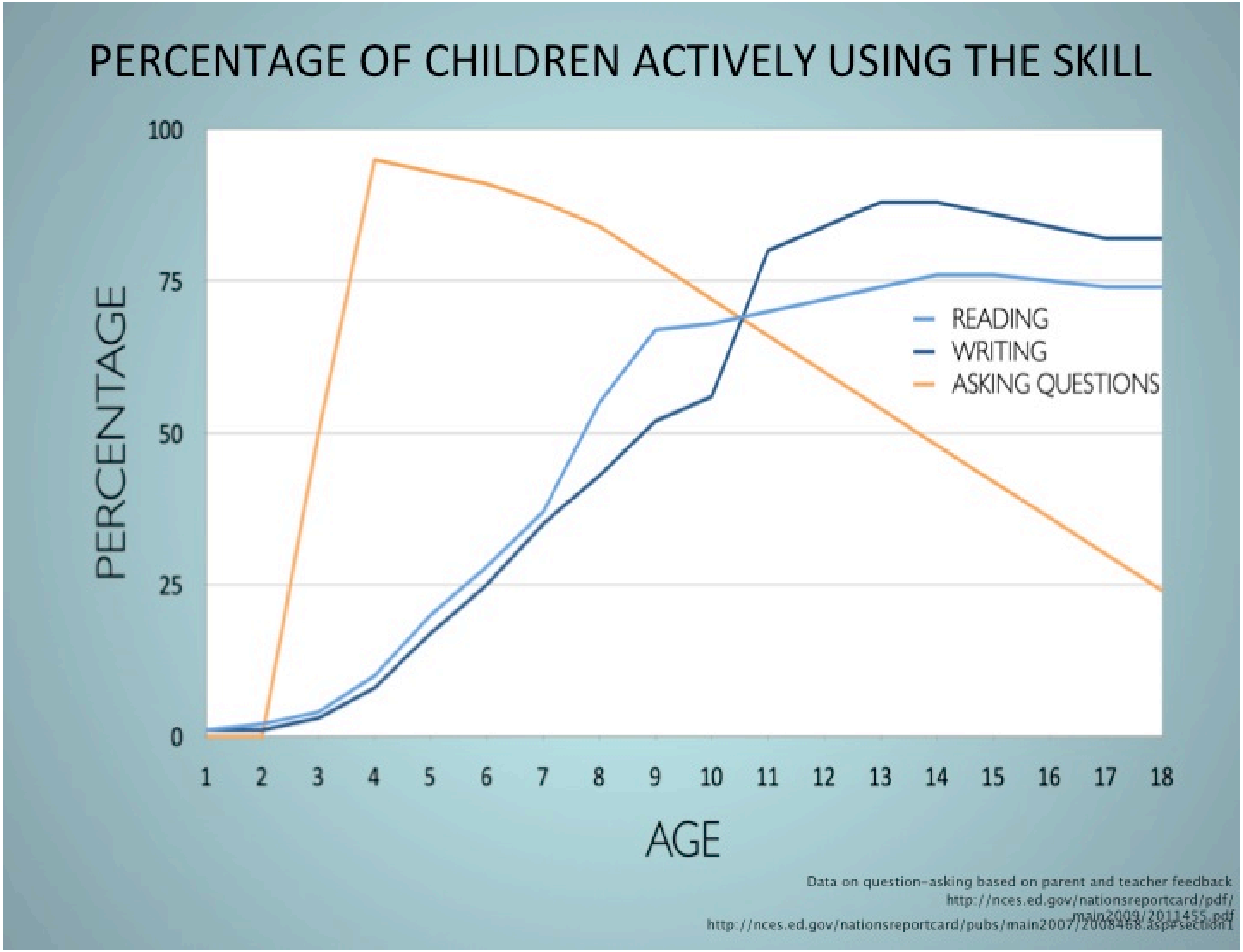
How does answer change as volume changes?

Cirillo, M., Pelesko, J.A., Felton-Koestler, M., & Rubel, L. (2016). Perspectives on Modeling in School Mathematics in C. Hirsch (Ed.) *Mathematical Modeling and Modeling Mathematics*, pp. 3-15, NCTM: Reston, VA.



1. Mathematical modeling *authentically* connects to the real world, starting with ill-defined, often messy real-world problems with no unique correct answer.
2. Mathematical modeling is used to explain phenomena in the real world and/or make predictions about future behavior of a system in the real world.
3. Mathematical modeling requires the modeler to be creative and make choices, assumptions, and decisions.
4. Mathematical modeling is an iterative process.
5. There are multiple paths open to the mathematical modeler, and no one, clear “right” approach or answer.

“Where do I even start?”

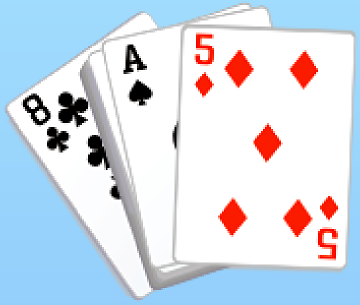

PERCENTAGE OF CHILDREN ACTIVELY USING THE SKILL
100
75
50
25
0
PERCENTAGE
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
AGE
READING
WRITING
ASKING QUESTIONS
Data on question-asking based on parent and teacher feedback
http://nces.ed.gov/nationsreportcard/pdf/main2009/2011455.pdf
http://nces.ed.gov/nationsreportcard/pubs/main2007/2008468.asp#section1

"Pseudocontext" versus Real World Problems

(Boaler, 2008, pp. 51-53)

Melissa and Joe are playing a game with complex numbers. If Melissa has a score of 5-4i and Joe has a score of 3+2i, what is their total score?

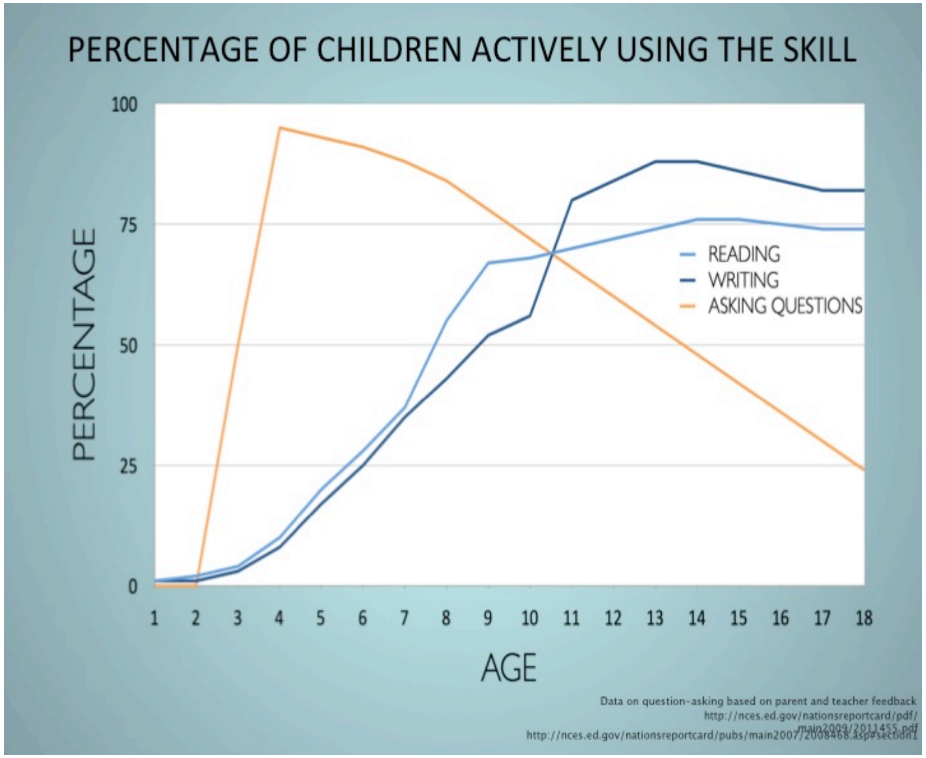
1. 8 + 6i
2. 8 + 2i
3. 8 – 6i
4. 8 – 2i



(NYS Regents, June, 2001, #11)

Learning without Reality

- To do well in math class, children know that they have to suspend reality and accept the ridiculous problems they are given.
- They know that if they think about the problems and use what they understand from life, they will fail.
- Over time, schoolchildren realize that when you enter Mathland you leave your common sense at the door.



- *The Question Formulation Technique*

(Rothstein & Santana, 2013)

USING STUDENT QUESTIONS

Students can use their questions for many purposes, including the following:

- Conduct Research
- Reports
- Conduct Experiments
- Independent Projects
- Write Papers/Essays
- Group and Individual Projects
- Socratic Seminars/Debates
- Prepare for Presentations/Interviews
- **ENGAGE IN MATHEMATICAL MODELING!**

COMPONENTS OF THE QUESTION FORMULATION TECHNIQUE™

1. **The Question Focus (QFocus)**
2. **The Rules for Producing Questions**
3. **Producing Questions**
4. **Categorizing Questions**
5. **Prioritizing Questions**
6. **Next Steps**
7. **Reflection**

The Question Focus (Qfocus)

- The Question Focus (QFocus) is usually developed by the teacher and serves as the jumping off point for student questions.
- It should be brief and provoke or stimulate new lines of thinking.
- **The QFocus should not be in the form of a question!**

Rules for Producing Questions

- The rules for producing questions must be discussed in advance and understood by the students. The rules for producing questions are:
  - Ask as many questions as you can
  - Do not stop to answer, judge, or discuss the questions
  - Write down every question *exactly* as it is stated
  - Change any statement into a question.
- Post or distribute the Rules for Producing Questions.

Rules for … uestions

- The rules for p… must be discussed in ac… od by the students. The … uestions are:
  - Ask as many c…
  - Do not stop t… ss the questions
  - Write down e… s it is stated
  - Change any st… n.
- Post or distribu… ducing Questions.

The Question Formulation Technique™ (QFT™)

Rules for Producing Questions

- Ask as many questions as you can.
- Do not stop to discuss, judge or answer any question.
- Write down every question EXACTLY as it is stated.
- Change any statement into a question.

Producing Questions

- After discussing the rules, divide students into groups of 3-5 students.
- Ask each group to identify a note-taker.
- Distribute or reveal the QFocus.
- **Students should produce as many questions as possible in the allotted time (start with 10 minutes; decrease time as students master the technique).**
- The rules should be followed, and the questions should be numbered.
- The note-taker should contribute questions.

Categorizing Questions

- Questions should be categorized as closed or open-ended questions:
  - **Closed-ended Questions** can be answered with a “yes” or “no” or with a one-word answer.
  - **Open-ended Questions** require more explanation.
- Students should label the questions with a “C” or an “O” and practice changing closed questions to open questions and vice-versa.

Prioritizing Questions

- Following criteria set by the teacher, students should prioritize their questions and choose three questions.
- Examples of criteria include questions that: most interest you, you consider to be the **most important, will best help you solve a problem, or you need/want to answer first.**

Sharing Questions & Next Steps

- Students can be asked to share:
  - Their three priority questions
  - Their reasons for choosing the priority questions
  - The numbers of the priority questions in the sequence of the entire list.
- Students should use their questions to conduct research or

Explore modeling tasks!

Reflection

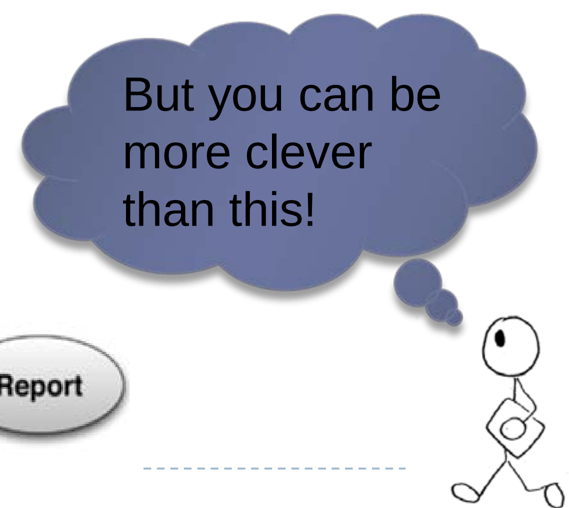
- Students should name what they have learned, how they learned it, and how they will use what they learned.

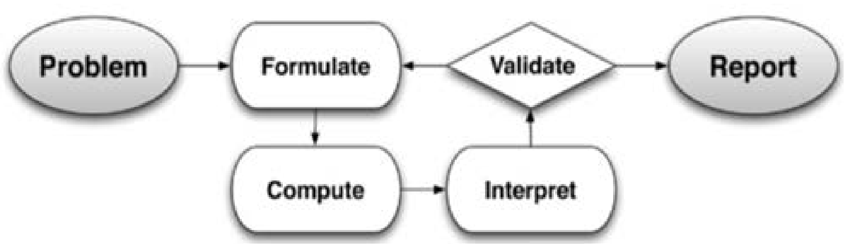
COMPONENTS OF THE QUESTION FORMULATION TECHNIQUE™

1. **The Question Focus (QFocus)**
2. **The Rules for Producing Questions**
3. **Producing Questions**
4. **Categorizing Questions**
5. **Prioritizing Questions**
6. **Next Steps**
7. **Reflection**

Some (generic) ways to think about the QFT specifically focused on mathematical modeling

- Real World → Problem: Questions amenable to a Mathematical Modeling approach
- Problem → Formulate: Questions that help you formulate a model (e.g., what factors affect X)
- Formulate → Analyze: Questions about your model
- Analyze → Interpret: Questions your model can answer
- Interpret → Validate: Questions to check your model against the real world





Core Plus (Course 3) Unit 6

9 At every location on Earth, the number of hours of daylight varies with the seasons in a predictable way. One convenient way to model that pattern of change is to measure time in days, beginning with the spring equinox (about March 21) as $t = 0$. With that frame of reference, the number of daylight hours in Boston, Massachusetts is given by $d(t) = 3.5 \sin \frac{2\pi}{365}t + 12.5$.

a. What are the amplitude, period, and frequency of $d(t)$? What do those values tell about the pattern of change in daylight during a year in Boston?

b. What are the maximum and the minimum numbers of hours of daylight in Boston? At what times in the year do they occur?

c. If the function giving the number of daylight hours in Point Barrow, Alaska, had the form $f(t) = a \sin bt + c$, how would you expect the values of a, b, and c to be related to the corresponding numbers in the rule giving daylight hours in Boston?

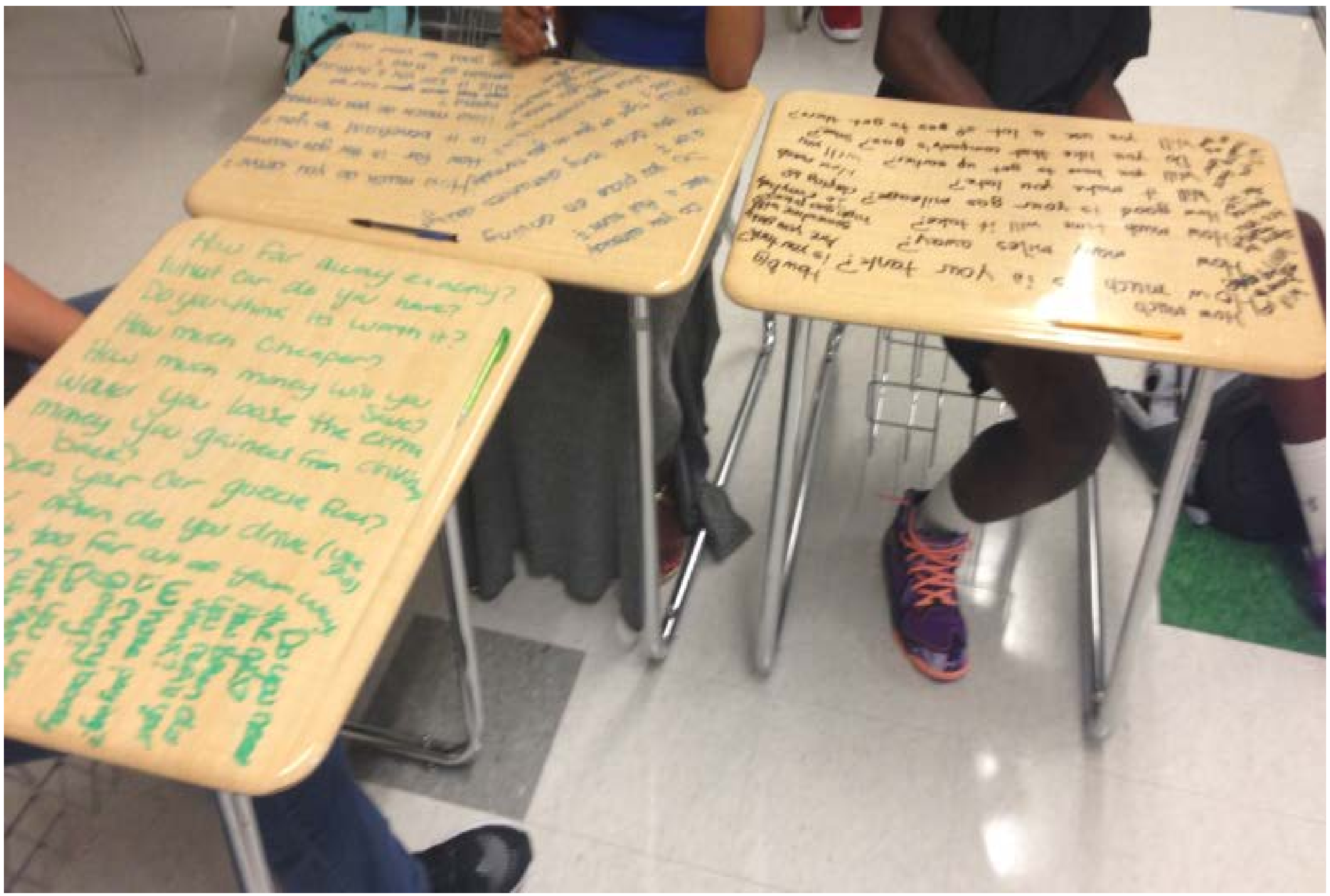
d. Why does it make sense that the function giving daylight hours at points on Earth should involve the circular function $\sin t$?

QFocus

Length of Day at Various Latitudes (in Hours and Minutes on the 15th of Each Month)

MONTH	EQUATOR	10°	20°	30°	40°	50°	60°	70°	80°	Poles	Month
January	12:07	11:35	11:02	10:24	9:37	8:30	6:38	0:00	0:00	0:00	July
February	12:07	11:49	11:21	11:10	10:42	10:07	9:11	7:20	0:00	0:00	August
March	12:07	12:04	12:00	11:57	11:53	11:48	11:41	11:28	10:52	10:52	September
April	12:07	12:21	12:36	12:53	13:14	13:44	14:31	16:06	24:00	24:00	October
May	12:07	12:34	13:04	14:22	15:22	17:04	22:13	24:00	24:00	24:00	November
June	12:07	12:42	13:20	14:04	15:00	16:21	18:49	24:00	24:00	24:00	December
July	12:07	12:40	13:16	13:56	14:49	15:38	17:31	24:00	24:00	24:00	January
August	12:07	12:28	12:50	13:16	13:48	14:33	15:46	18:26	24:00	24:00	February
September	12:07	12:12	12:17	12:23	12:31	12:42	13:00	13:34	15:16	24:00	March
October	12:07	11:55	11:42	11:28	11:10	10:47	10:11	9:03	5:10	0:00	April
November	12:07	11:40	11:12	10:40	10:01	9:06	7:37	3:06	0:00	0:00	May
December	12:07	11:32	10:56	10:14	9:20	8:05	5:54	0:00	0:00	0:00	June

Students' Questions



Students' Questions

- Could you make a graph out of this data?
- What factors made length of days vary?
- Why do the numbers decrease as the degrees increase?
- Is there a pattern between the degrees and length of day?
- Does longitude affect it?
- Why are all the times in the equator column the same?
- Why is it only latitude? Could longitude affect the data?

The New Task

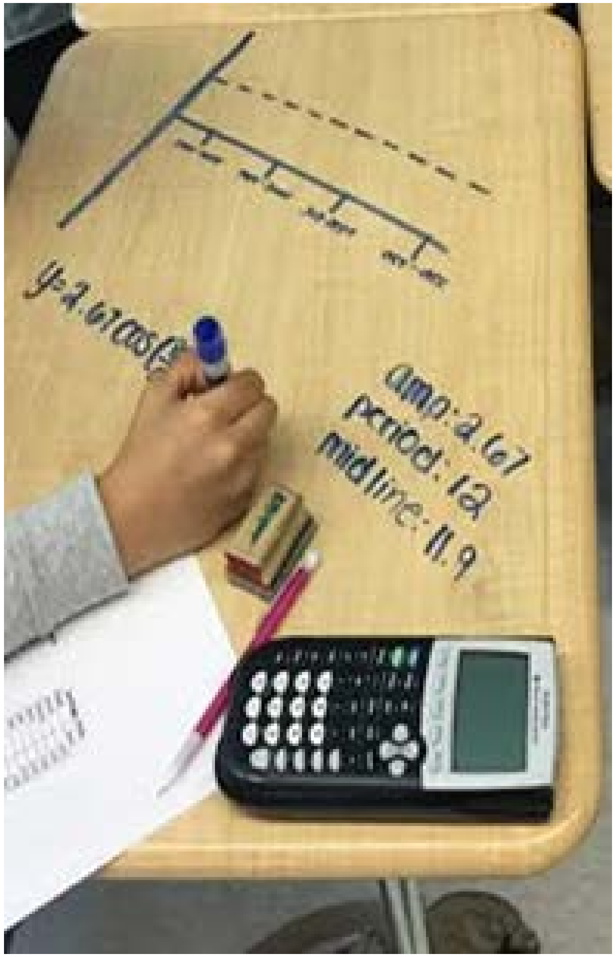
Middletown, Delaware

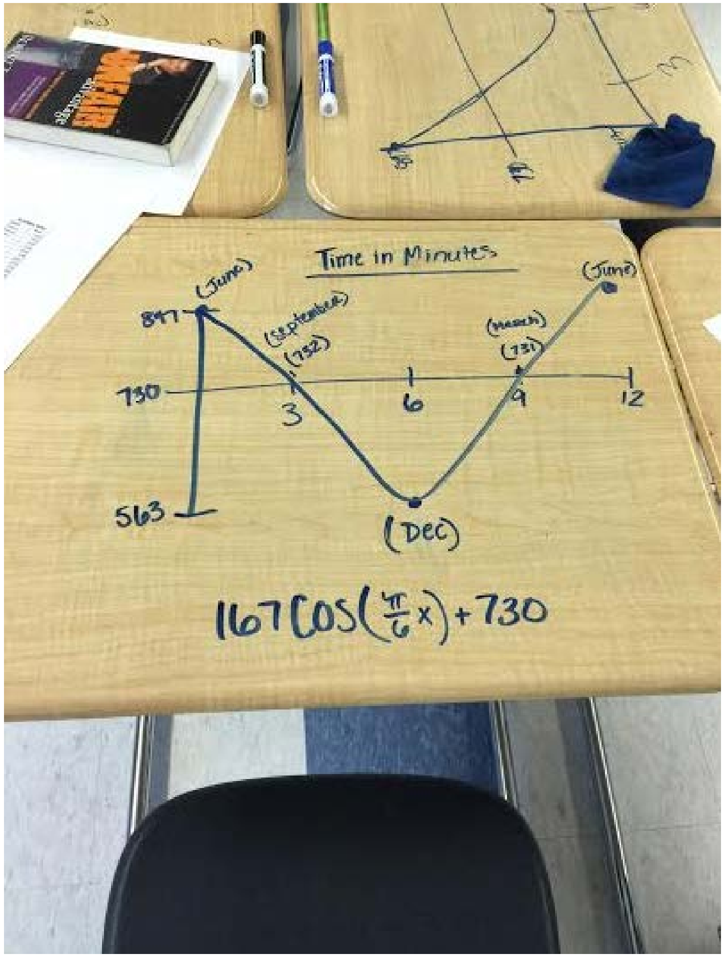
Month	Hours
January	9:52
February	10:59
March	12:11
April	13:29
May	14:31
June	14:57
July	14:32
August	13:29
September	12:12
October	10:56
November	9:50
December	9:23

Santiago, Chile

Month	Hours
January	13:57
February	13:03
March	12:06
April	11:05
May	10:18
June	9:58
July	10:17
August	11:05
September	12:05
October	13:05
November	13:58
December	14:20

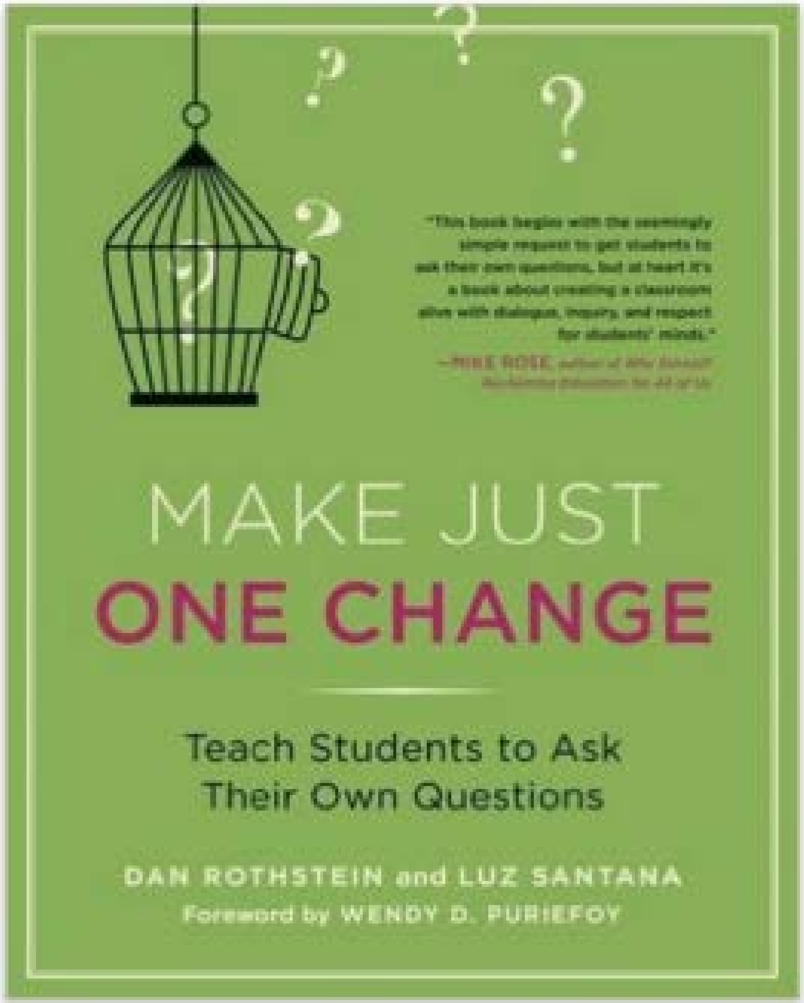
Student Work

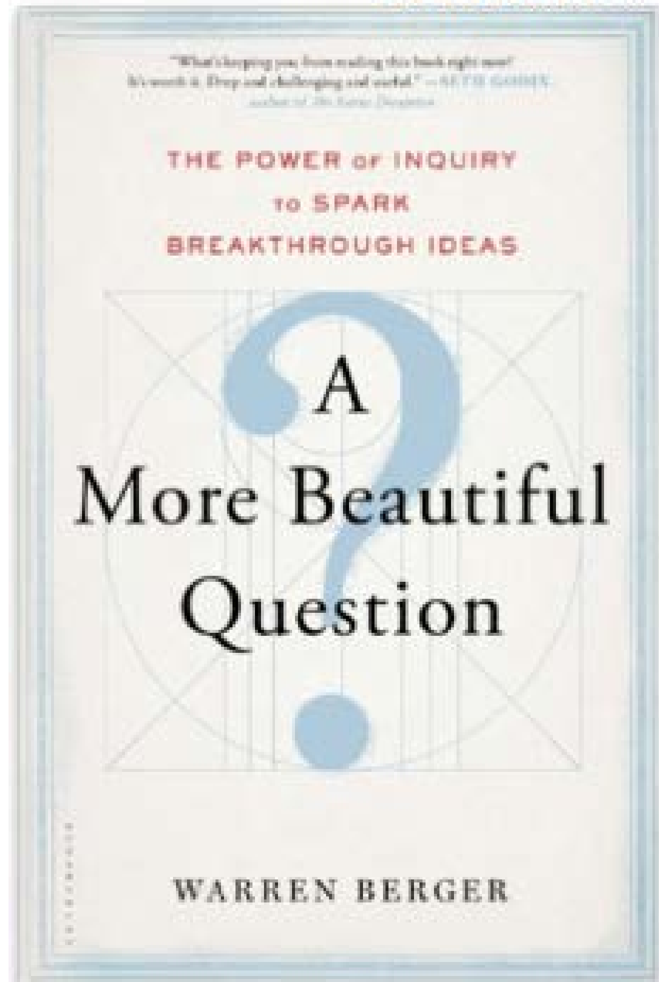




Motivating Students

- "The second time we used the QFT…students were again coming up with more and questions. They were writing on their neighbors' desks because they had run out of room on their own!...I honestly think that the QFT has done great things for my classes."





www.rightquestion.org

Central Message

- To successfully implement the Mathematical Modeling Standards in CCSSM, mathematics educators at all levels need to understand what Mathematical Modeling is, and how it is done in practice.
- It is only through this understanding that we will be able to support students in Modeling with Mathematics.

Final Take-Aways

- Expand content knowledge
- Develop deep partnerships between mathematical modelers and the math education community
- Address misconceptions about mathematical modeling
- Read pp. 72-73 with teachers at all levels

Resources

- *Mathematical Modeling Handbook* **(Comap, 2012)**
- *GAIMME Report* (SIAM & NCTM, Forthcoming)
- *Mathematical Modeling & Modeling Mathematics* (NCTM, 2016)
- *Math Modeling: Getting Started and Getting Solutions* (SIAM, Bliss et al., 2014)
- *Next Generation Science Standards*

Questions?

- Check out our blog:

  www.modelwithmathematics.com

- Check out our book:

  Model with Mathematics

  forthcoming from Math Solutions

mcirillo@udel.edu, pelesko@udel.edu

Twitter:@peleskoj

Rate this conference presentation on the NCSM App by answering four short questions.

Download available presentation handouts from the Online Planner on the App and/or after the conference from our website (look for Speaker Handouts and PPTs under Conference Essentials).

Join the conversation! Tweet us using the hashtag #NCSM2016